Frege's *Habilitationsschrift* and the Functional Approach to Magnitude. A Neglected Precursor of Computability

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Computing theory, together with the logical perspective associated to it, and the philosophy that could be built upon its postulates, appear as naturally diverging from (if not openly opposed to) those of the logicist and set-theoretical tradition of Frege, Russell or Carnap. The reason for this is certainly to be found in the fact that Gödel's incompleteness famous results of 1931 provided a decisive argument against the foundational aspirations of logicism, renewing a concern with the problem of "effective calculation", all of which lay at the basis of fundamental results in the theory of computability (such as the perfection of Church's Lambda-Calculus as a universal programming language and the formulation of Church-Turing thesis). As a consequence, computer science is more likely to recognise a precursor in Dedekind and Peano, or even in Babbage or Boole, than in the work of Gottlob Frege.

However, if closer attention is payed to Frege's entire work, his approach to logic and formal systems proves to be closer in some essential respects to the computational approach than to the logicist tradition as it was later on forged by Russell and his followers. For if a number of typical logicist postulates are indeed present in Frege's best-known publications, an original connection between functionality and arithmetic underlies his entire work, which implies an alternative conception of logic. The main reason for this is that Frege's functional perspective, unlike Dedekind's o Russell's for example, shows to be both historically and theoretically independent from an approach in terms of classes or sets.

The connection between Frege's functional approach to arithmetic an computability can be better grasped if attention is payed to his early development of a theory of functional iteration meant to inform a concept of magnitude and number comprehensive enough to deal with all the quantities that cannot be immediately conceived as objects of intuition. The development of such a theory is the main purpose of Frege's *Habilitationsschrift*, written in 1874, significantly intituled "*Methods of Calculation based on an Extension of the Concept of Quantity*", and remained surprisingly neglected and understudied by both Frege scholars and historians of computability.

In these pages, Frege strives for the creation of a concept of quantity out of the successive iterations of an operation (*ff*, *fff*,...) acting on an object. The author intends to achieve this task by the definition of the "quantity of a function", associated to the relation between multiple iterations of a function, in such a way that, e.g. the functions ($\phi(\phi(x))$ and $\phi(\phi(\phi(x)))$ are assigned respectively double or triple the quantity of $\phi(x)$, the function $\psi(x)$ a fourth of the quantity of $\phi(x)$ if $\phi(x)=\psi(\psi(\psi(\psi(x))))$, the quantity of $\chi(x)$ being the reciprocal of that of $\phi(x)$ if $\phi\chi(x)=x$, and the null quantity being assigned to the function that x is of itself.

Once the concept of quantity is thus defined in terms of functional iteration, the question arises of the determination of the belonging of given functions to the same "quantitative domain" as well as of the quantitative relation between functions of the same quantitative domain. The treatment of this fundamental problem leads Frege to define a general form of quantitative functions by means of the general functional equations:

$$f(1, x)=f(x),$$

$$f(n_0, f(n_1, x))=f(n_0+n_1, x),$$

in which recursion suggests itself as an equational schema that organises the functional nature of an iterative approach to numbers. The variable x ranging over real values, Frege's solution to this equation relies on the definition of "quantitative equations", in which the quantity n associated to a function is expressed in terms of the argument x and the value X of the function $f(n, x)$, i.e. $n_0=\psi(X, x_0)$. This gives rise, after elimination and substitution, to the general quantitative equation:

$$\psi(X, x_0)+ \psi(x_0, x_1)= \psi(X, x_1)$$

thanks to which, Frege is capable of identifying formal conditions for given numerical (real-valued) functions to be part of a quantitative domain, (namely, that their quantities are related as the logarithm of their differential quotients at a fixed point $f(x_1)=x_1$), as well as to establish (equational) methods of calculation to determine quantitative relations between functions.

If these early formulations constitute the first steps of what would later be called "iteration theory", they can be understood as bearing a deeper sense in the framework of Frege's project, viz. that of providing a general concept of number independent from intuition by means of an original approach to recursion over a purely functional domain. What is more, it can be shown that this particular approach to the general concept of number underlies Frege's entire logical undertaking. Indeed, I'll claim that this conception informs Frege's design of the propositional function as well as of the logical system built upon it: the propositional function would be nothing more than the result of the necessary symbolical or semiological generalisation of numerical functions, on which recursion can freely operate in order to provide a general and non-intuitive concept of number.

Such a system can only incidentally be associated with the set-theoretical conception of Cantor, Dedekind, Peano or Russell. In particular, Frege's approach can be distinguished from Dedekind's famous formulations in that the latter relies on a primitive concept of "thing" and "system" (practically equivalent to those of element and set), which ties the recursive construction of the natural numbers to a set-theoretical interpretation. Unrestricted by this ontological or semantical condition, Frege's purely functional perspective is otherwise attached to a semiological approach to functional iteration. From this point of view, recursion does not appear as an operation on objects, but rather as a schema identifying iterative processes in a given dynamics of signs. Hence the need to develop a conceptual *notation* (i.e. the *Begriffsschrift*) – rather than a theory of sets or classes – in which signs are entirely organized as system of functional articulation upon which a principle of induction can be built in order to define a logical concept of number. If Frege's later adoption of an extensional view restricted his functional language to an interpretation in terms of classes, Russell's paradox will call into question this restriction, revealing the intimate relation between Frege's conceptual notation and Church's Lambda-Calculus as a universal computing language, as shown by the correspondence between Frege and Russell.

All of these circumstances, together with the later attempts of Church to build a Logic of Sense openly inspired by Frege, allow to consider Frege as a strong and insightful pioneer of the logical and philosophical stakes associated to the tradition of computing sciences, and to contribute to the conceptual basis of a computational approach to logic.

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