

# In Search of the Roots of Formal Computation.

## A Summary

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### Kantian background of formal computation

It became clear by the 1930s that steps of formal computation are also steps of formal deduction as defined by recursion equations and other principles of arithmetic. This development began with attempts at giving a foundation to the laws of commutativity and associativity of sum: Followers of Kant's doctrine of the *synthetic a priori* in arithmetic, a certain Johann Schultz as foremost, missed by a hair's breadth the proper recursive definition of addition that appeared instead first in a book of Hermann Grassmann of 1861. Schultz in his book *Anfangsgründe der reinen Mathesis* (Basics of pure mathesis, 1790) missed the inductive proofs of commutativity and associativity, and could not do better than claim that any attempted proof of the commutativity of addition would be circular. He gives instead an inductive proof of the commutativity of product in which the right recursion equations appear as "corollaries", a reverse of the conceptual order recursive definition – inductive proof. This order was found by Grassmann and a line can be followed from it to Hankel, Schröder, Dedekind, Peano, and Skolem, the last mentioned marking the birth of *recursive arithmetic*.

Schultz in his two-volume *Prüfung der Kantischen Kritik der reinen Vernunft* (Examination of the Kantian critique of pure reason,

1789 and 1792) asks (vol. 1, p. 220):

How do I know, then, that this willful procedure [of changing the order of a sum] that lies in no way in the concept of addition itself, provokes no difference in the sum  $7+5$ ?

Schultz gives a proof of the equation  $7+5=1$  – perhaps the only formula in Kant's *Kritik*, a "smoking gun" of sorts in later writers. He uses the axioms of commutativity and associativity of addition, then states that they are "indispensable for arithmetic" (p. 219), for without them, one would get only that  $7+5=7+(4+1)=7+(1+1+1+1)$ . I take this to be Schultz' reading and interpretation of Kant passage. With commutativity and associativity taken into use, one gets the following proof of the Kantian formula (*Prüfung*, p. 220):

Instead of all of 5, or its units taken together and added at once to 7, I must take them instead successively one by one, and instead of  $7+(4+1)$  first set  $7+(1+4)$ , and in place of it  $(7+1)+4$ , so then I get thanks to the concept of the number 8, namely that it is  $7+1$ , first  $8+4$  i.e.  $8+(3+1)$ . Instead of this I have to set again  $8+(1+3)$ , and  $(8+1)+3$  in place of

it, so I get  $9+3$ , i.e.,  $9+(2+1)$ . Setting for this  $9+(1+2)$ , and therefore  $(9+1)+2$ , gives  $10+2$ , i.e.,  $10+(1+1)$ , and setting for this  $(10+1)+1$  gives finally  $11+1$ , i.e. 12. Every arithmetician knows that this is the only way through which we can come to the insight that the theorem is correct.

Here is a clear sign of awareness of the recursive definition of sum. Schultz missed by a hair's breadth the correct recursion equation, as can be seen by the comparison:

Schultz 1790:  $7+5 = 7+(4+1) = 7+(1+4) = (7+1)+4$

Grassmann 1861:  $7+5 = 7+(4+1) = (7+4)+1$

## Grassmann

Hermann Grassmann, a high-school teacher, had the happy idea of applying the recursive procedure, as found in the combinatorics of the early 19<sup>th</sup> century, to the most elementary parts of arithmetic, namely the basic arithmetic operations. His 1861 *Lehrbuch der Arithmetik für höhere Lehranstalten* contains the first explicit recursive definitions of arithmetic operations, ones that go hand in hand with inductive proofs of properties of the recursively defined operations. Grassmann's definition of addition uses a unit denoted by  $e$  (p. 4):

15. *Explanation.* If  $a$  and  $b$  are arbitrary members of the basic sequence, one un-

derstands with the sum  $a+b$  that member of the basic sequence for which the formula  $a+(b+e)=a+b+e$  holds.

He thus missed the inductive proofs of commutativity and associativity, and could not do better than claim that any attempted proof of the commutativity of addition would be circular (p. 221). In the *Anfangsgründe*, Schultz had declared the commutativity of addition to be "immediately evident" (p. 42).

Remarkably, when Schultz comes to prove the commutativity of product by induction, in his *Kurzer Lehrbegriff der Mathematik* (Short course in mathematics, 1797), he uses the correct recursion equations, called "corollaries" (p. 36).

The recursive definition of sum is put into use in Grassmann's "inductory" (inductorisch) proofs of the basic properties of addition, such as associativity and commutativity. Next product is defined similarly. A comparison to Schultz may be useful: The first step in the proof was  $n \cdot 2 = n \cdot (1+1) = n \cdot 1 + n \cdot 1$ , now an instance of Grassmann's recursive definition of product by the equation  $a \cdot (b+1) = a \cdot b + a$ .

$$A+(B+1)=(A+B)+1.$$

Hankel now states that "this equation determines every sum" and shows how it goes: By setting  $B=1$  in the equation one has  $A+2=A+(1+1)=(A+1)+1$ , and with  $B=2$  one has  $A+3=A+(2+1)=(A+2)+1$  so that  $A+2$  and  $A+3$  are numbers in the sequence of integers

## The reception of Grassmann's idea

Grassmann's approach to the foundations of arithmetic is found explained in detail in the first of Hermann Hankel's two-volume treatise *Vorlesungen über die complexen Zahlen und ihre Functionen* of 1867. The sum  $(A+B)$  of two numbers is defined as in Grassmann's recursion equation (p. 37):

(p. 37):

*In this way one finds through a recurrent procedure, one that goes on purely mechanically without any intuition, unequivocally every sum of two numbers.*

Grassmann's approach is next described by Ernst Schröder in his 1873 *Lehrbuch der Arithmetik und Algebra*. The book contains an introduction and chapters on arithmetic operations with a presentation that follows directly Hankel's divisions. Grassmann's recurrent mode of counting sums is explained through detailed examples (pp. 63–64):

(5)  $2=1+1, 3=2+1, 4=3+1, 5=4+1$ , etc.,

The natural numbers are hereby defined recurrently. Namely, to give in a complete

way the meaning of a number, i.e., to express it through the unity, one has to go back from it to the previous number and to run through backwards (recurre) the whole sequence.

Schröder's presentation of how the natural numbers are generated by the  $+1$ -operation is described through a notational novelty (p. 64):

If  $a$  is a number from our sequence: 1, 2, 3, 4, 5,... then even  $a+1$  is one, namely  $a' = a+1$  is the general form of equations (5).

This seems to be the first place in which the successor operation obtains a separate notation, one that became later the standard one. It was a conceptually important step and the notation was used by Dedekind in 1888.

## Peano

Peano writes in his 1889 *Arithmetices Principia, Nova Methodo Exposita* that he has followed in logic amongst others Boole, and for proofs in arithmetic the book by Grassmann (1861) [in arithmeticae demonstrationibus usum sum libro: H. Grassmann]. Among his references there is also the book of Schröder of 1873. Sum is defined and explained by:

18.  $a, b \in N. \supset a+(b+1) = (a+b)+1$ .

*Note.* This definition has to be read as follows: if  $a$  and  $b$  are numbers, and if  $(a+b)+1$  has a sense (that is, if  $a+b$  is a number) but  $a+(b+1)$  has not yet been defined, then  $a+(b+1)$  signifies the number that follows  $a+b$ .

When introducing his primitive signs for arithmetic, Peano enlisted *unity*, notation 1, and a *plus* 1, notation  $a+1$ . Thus, the sum of two numbers was not a basic notion, but

just the successor, and definition 18 laid down what the addition of a successor  $b+1$  to another number means, in terms of his primitive notions. Quite amazingly, Frege, and following him Jean van Heijenoort who edited Peano's treatise in English, insisted that Grassmann's and Peano's recursive definition is "circular" and therefore defective. The matter was put in correct light by Paul Bernays, in the first volume of the *Grundlagen der Mathematik* of 1934 (p. 219):

It is customary in mathematics to give the process of continuation [of the number series] through " $+1$ ". This kind of notation has, however, the defect that the conceptual distinction between taking " $a+1$ " as the number that follows  $a$  and on the other hand as the sum of  $a$  and 1 fails to be represented.

# Clouds and Clocks

## Some Historical Thoughts About Time in Computer-assisted Music Composition (1955-2015)

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All human activities take place over time (Mainzer), each with its own temporality which itself evolves over time (Hartog).

This paper is part of a historical reflection (Leduc) on the way that temporality is articulated in music and in computer technology, two domains whose temporal logics are initially very different (Kramer, Berry, Pressing)

More specifically, this paper focuses on my specialism: the practices of composers of academic computer-assisted music. It concentrates on the temporal aspects of the act of composition.

An initial phase, beginning around 1955 (Lazzarini, Baudouin, Viel), prolonged the reflections on composition of the period 1900–1950 (Donin). The explosion of calculation

speeds in the 1980s allowed “real time” to emerge (Manoury), followed since the beginning of the 2000s by “live” composition time. This has brought computer time and musical time closer and closer together. The research currently being carried out in France, whether in computer science (Berry) or in musical research (Desainte-Catherine, Janin, Assayag) bears witness to convergence and cross-fertilisation between the reflections on time proper to these two fields.

The new types of music that are currently emerging subvert the “traditional” composition process. This movement is part of the general crisis affecting the futurist regime of historicity, and illustrates the growing influence of presentism.

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